Closed-Form Expressions for a Number of Fourier-Bessel Series Encountered in the GSD Method

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Abstract—**In the application of the generalized spectral-domain (GSD) method to the analysis of waveguides, infinite sums over eigenmodes have to be evaluated. The efficiency of the GSD method is considerably enhanced if these sums can be replaced by closed-form expressions. Up to now, appropriate identities are known for circular and rectangular waveguides only. Expressions associated with sector waveguides will be considered.**

I. Introduction

In high output power gyrotrons, the interaction between the electron beam and the electromagnetic field takes place in a highly overmoded cavity. Therefore, mode competition becomes a severe problem especially in the design of gyrotrons operating at higher harmonics of the cyclotron frequency. Recently, it has been suggested to overcome this problem using a new cavity structure [1]. Fig. 1 shows the cross section of the new cavity.

In [1], a conventional field matching method has been applied to analyze the composite waveguide. But this method is not suitable for this structure because the expansion functions do not fulfill the edge conditions at the slot edges. This leads to oversized matrices, slow convergence, and inaccurate field distributions.

This problem can be circumvented by applying the GSD method. It is based on short-circuiting the coupling slots and replacing the nonvanishing slot tangential electric field by two surface magnetic currents at the two sides of the short circuit. The two surface currents that are yet unknown are equal in magnitude and opposite in sign. The electromagnetic field in the individual waveguides is expanded with respect to the corresponding eigenmodes. Applying the moment method, the unknown surface magnetic currents are expanded in terms of suitable basis functions that satisfy the edge conditions at the slot edges. Testing the continuity of the tangential magnetic field across the slots by the same basis functions (Galerkin's procedure) results in a matrix equation, the unknowns of which are the expansion coefficients of the surface magnetic currents.

Each matrix element contains a doubly infinite sum that has to be summed with respect to the azimuthal and radial indices corresponding to the eigenmodes of the circular and sector waveguides. For the structures considered in [2], it has been found that the summations over the indices corresponding to the directions normal to the surface currents have closed-form expressions. This increases the numerical efficiency of the GSD method drastically.

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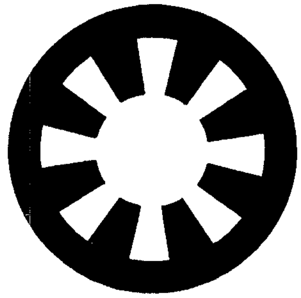

Fig. 1. Cross section of the new gyrotron resonator.

In [3], closed-form expressions for Fourier and Fourier–Bessel series have been given. Following the analysis presented in [3], the eigenmodes of one waveguide are expanded with respect to the eigenmodes of another waveguide. Use of the orthogonality property of the eigenmodes is then made in order to yield some identities. The closed-form expressions which we are looking for can consequently be obtained by suitable linear combinations of these identities.

II. Basic Formulation

Let us consider two waveguides of cross section $A^{(1)}$ and $A^{(2)}$, with $A^{(1)} \subset A^{(2)}$, which are joined in the plane $z =$ constant, where z denotes the axial coordinate.

Let $\{h_{zn}^{(i)}\}$ and $\{e_{zn}^{(i)}\}$ $(i = 1, 2)$ be the complete sets of axial magnetic and axial electric fields in the individual waveguides. It is important to note that the set $\{h_{zn}^{(i)}\}$ is complete only if $h_{z0}^{(i)}$ is included. The mode characterized by $h_{z0}^{(i)}$ has only an axial magnetic field which is constant over the cross section of the waveguide. The corresponding cut-off wavenumber must vanish. For the axial electric and axial magnetic fields, the following orthogonality relations hold:

$$\int_{A^{(i)}} e_{zm}^{(i)} e_{zn}^{(i)*} \, dS = \frac{1}{\left(k_m^{e(i)}\right)^2} \delta_{mn} \tag{1a}$$

$$\int_{A^{(i)}} h_{zm}^{(i)} h_{zn}^{(i)*} \, dS = \frac{1}{\left(k_m^{h(i)}\right)^2} \delta_{mn} \tag{1b}$$

$$\int_{A^{(i)}} h_{z0}^{(i)} h_{zn}^{(i)*} \, dS = A^{(i)} \delta_{0n}, \tag{1c}$$

where $k_m^{h(i)}$ and $k_m^{e(i)}$ are the cut-off wavenumbers of the mth TE and the mth TM eigenmodes, respectively. The asterisk $(*)$ denotes complex conjugate and δ_{mn} is the Kronecker delta. Equation (1c) fixes $|h_{z0}^{(i)}|$ to unity.

The transverse fields of the eigenmodes satisfy the orthogonality relations

$$\int_{A^{(i)}} \nabla_t e_{zm}^{(i)} \cdot \nabla_t e_{zn}^{(i)*}\, dS = \delta_{mn} \tag{1d}$$

$$\int_{A^{(i)}} \nabla_t h_{zm}^{(i)} \cdot \nabla_t h_{zn}^{(i)*}\, dS = \delta_{mn} \tag{1e}$$

$$\int_{A^{(i)}} \left(\nabla_t e_{zm}^{(i)} \times \nabla_t h_{zn}^{(i)*}\right) \cdot \hat{\boldsymbol{k}}\, dS = 0, \tag{1f}$$

where ∇_t and $\hat{\boldsymbol{k}}$ are the transverse component of the del-operator and the unit vector in axial direction, respectively. The sets $\{\nabla_t e_{zm}^{(i)}\}$ and $\{\nabla_t h_{zm}^{(i)} \times \hat{\boldsymbol{k}}\}$ are complete with respect to curl-free and divergence-free transverse electric fields, respectively, which can be supported by the ith waveguide. The transverse and axial components of the eigenmodes corresponding to waveguide (2) can be expanded with respect to the eigenmodes of waveguide (1) on the common cross section $A^{(1)}$:

$$\nabla_t e_{zm}^{(2)} = \sum_p^{\infty} r_{mp} \nabla_t e_{zp}^{(1)} + \sum_p^{\infty} s_{mp}\left(\nabla_t h_{zp}^{(1)} \times \hat{\boldsymbol{k}}\right) \tag{2a}$$

$$\nabla_t h_{zm}^{(2)} \times \hat{\boldsymbol{k}} = \sum_p^{\infty} t_{mp} \nabla_t e_{zp}^{(1)} + \sum_p^{\infty} u_{mp}\left(\nabla_t h_{zp}^{(1)} \times \hat{\boldsymbol{k}}\right) \tag{2b}$$

$$e_{zm}^{(2)} = \sum_p^{\infty} v_{mp} e_{zp}^{(1)} \tag{2c}$$

$$h_{zm}^{(2)} = w_{m0} h_{z0}^{(1)} + \sum_p^{\infty} w_{mp} h_{zp}^{(1)}. \tag{2d}$$

Making use of the orthogonality relations (1a)–(1f) results in

$$r_{mp} = \int_{A^{(1)}} \nabla_t e_{zm}^{(2)} \cdot \nabla_t e_{zp}^{(1)*}\, dS \tag{3a}$$

$$s_{mp} = \int_{A^{(1)}} \left(\nabla_t e_{zm}^{(2)} \times \nabla_t h_{zp}^{(1)*}\right) \cdot \hat{\boldsymbol{k}}\, dS \tag{3b}$$

$$u_{mp} = \int_{A^{(1)}} \nabla_t h_{zm}^{(2)} \cdot \nabla_t h_{zp}^{(1)*}\, dS \tag{3c}$$

$$v_{mp} = \left(k_p^{e(1)}\right)^2 \int_{A^{(1)}} e_{zm}^{(2)} e_{zp}^{(1)*}\, dS \tag{3d}$$

$$w_{mp} = \left(k_p^{h(1)}\right)^2 \int_{A^{(1)}} h_{zm}^{(2)} h_{zp}^{(1)*}\, dS \tag{3e}$$

$$w_{m0} = \frac{1}{A^{(1)}} \int_{A^{(1)}} h_{zm}^{(2)} h_{z0}^{(1)*}\, dS. \tag{3f}$$

Applying Stoke's theorem, it can be shown that t_{mp} vanishes. If we substitute the expansions (2a)–(2d) into the orthogonality relations (1d), (1e), (1a), and (1b) for waveguide (2) we get

$$\int_{A^{(2)}-A^{(1)}} \nabla_t e_{zm}^{(2)} \cdot \nabla_t e_{zn}^{(2)*}\, dS + \sum_p^{\infty} r_{mp} r_{np}^* + \sum_p^{\infty} s_{mp} s_{np}^* = \delta_{mn} \tag{4a}$$

$$\int_{A^{(2)}-A^{(1)}} \nabla_t h_{zm}^{(2)} \cdot \nabla_t h_{zn}^{(2)*}\, dS + \sum_p^{\infty} u_{mp} u_{np}^* = \delta_{mn} \tag{4b}$$

$$\int_{A^{(2)}-A^{(1)}} e_{zm}^{(2)} e_{zn}^{(2)*}\, dS + \sum_p^{\infty} \frac{1}{\left(k_p^{e(1)}\right)^2} v_{mp} v_{np}^* = \frac{1}{\left(k_m^{e(1)}\right)^2} \delta_{mn} \tag{4c}$$

$$\int_{A^{(2)}-A^{(1)}} h_{zm}^{(2)} h_{zn}^{(2)*}\, dS + A^{(1)} w_{m0} w_{n0}^* + \sum_p^{\infty} \frac{1}{\left(k_p^{h(1)}\right)^2} w_{mp} w_{np}^* = \frac{1}{\left(k_m^{h(1)}\right)^2} \delta_{mn}. \tag{4d}$$

Note that the TE mode with h_z = constant enters (4d) as $A^{(1)} w_{m0} w_{n0}^*$.

III. Sector Waveguide

In a sector waveguide with radii ρ_1 and ρ_2 ($\rho_1 < \rho_2$) and angle 2Θ, the axial electric and magnetic fields can be written as

$$e_{zmn} = N_{mn}^e F_{mn}^e(k_{mn}^e \rho) \sin(\nu_m(\varphi + \Theta)) \tag{5a}$$

$$h_{zmn} = N_{mn}^h F_{mn}^h(k_{mn}^h \rho) \cos(\nu_m(\varphi + \Theta)) \tag{5b}$$

where ρ and φ are the radial and azimuthal coordinates, respectively. N_{mn}^e and N_{mn}^h denote normalization quantities which are given by (5c) and (5d) below.

The prime (′) means the derivative of the corresponding function with respect to its argument. The functions

$$N_{mn}^e = \sqrt{\frac{2}{\Theta}} \frac{1}{\sqrt{(k_{mn}^e \rho_2)^2 \left(F_{mn}^{e\prime}(k_{mn}^e \rho_2)\right)^2 - \frac{4}{\pi^2}}} \tag{5c}$$

$$N_{mn}^h = \sqrt{\frac{2}{\Theta(1+\delta_{m0})}} \cdot \frac{1}{\sqrt{\left((k_{mn}^h \rho_2)^2 - \nu_m^2\right)\left(F_{mn}^h(k_{mn}^h \rho_2)\right)^2 - \left((k_{mn}^h \rho_1)^2 - \nu_m^2\right) \frac{4}{\pi^2 (k_{mn}^h \rho_1)^2}}}. \tag{5d}$$

$F^e_{mn}(k^e_{mn}\rho)$ and $F^h_{mn}(k^h_{mn}\rho)$ describing the radial variation of the fields read

$$F^e_{mn}(k^e_{mn}\rho) = N_{\nu_m}(k^e_{mn}\rho_1)J_{\nu_m}(k^e_{mn}\rho) - J_{\nu_m}(k^e_{mn}\rho_1)N_{\nu_m}(k^e_{mn}\rho) \quad (5e)$$

$$F^h_{mn}(k^h_{mn}\rho) = N'_{\nu_m}(k^h_{mn}\rho_1)J_{\nu_m}(k^h_{mn}\rho) - J'_{\nu_m}(k^h_{mn}\rho_1)N_{\nu_m}(k^h_{mn}\rho) \quad (5f)$$

where J_{ν_m} and N_{ν_m} are the Bessel and Neumann functions of order ν_m, respectively. The boundary conditions at $\varphi = \pm\Theta$ leads to

$$\nu_m = \frac{m\pi}{2\Theta}. \quad (5g)$$

k^e_{mn} and k^h_{mn} have to fulfill the characteristic equations

$$F^e_{mn}(k^e_{mn}\rho_2) = 0 \quad (5h)$$

$$F^{h\prime}_{mn}(k^h_{mn}\rho_2) = 0 \quad (5i)$$

Let us now consider a sector waveguide discontinuity. In the azimuthal direction both waveguides cover the same sector with an angle of 2θ. In the radial direction the waveguides (1) and (2) are characterized by the radii $\rho_1^{(1)} = a$, $\rho_2^{(1)} = b$ and $\rho_1^{(2)} = c$ $(c < a)$, $\rho_2^{(2)} = \rho_2^{(1)}$, respectively.

A. TE Modes

Exploiting the identities (4b) and (4c) yields

$$\left((1+\delta_{m0})\Theta N^{h(2)}_{mn} k^{h(2)}_{mn}\frac{2}{\pi}F^{h(2)\prime}_{mn}\left(k^{h(2)}_{mn}a\right)\right)^2 \cdot \sum_p^\infty \left(N^{h(1)}_{mp}\frac{k^{h(1)}_{mp}}{\left(k^{h(1)}_{mp}\right)^2 - \left(k^{h(2)}_{mn}\right)^2}\right)^2$$
$$= \left(k^{h(2)}_{mn}\right)^2 \int_{A^{(1)}} \left|h^{(2)}_{zm}\right|^2 dS - (1+\delta_{m0}) \Theta\left(N^{h(2)}_{mn}\right)^2 k^{h(2)}_{mn} a F^{h(2)}_{mn}\left(k^{h(2)}_{mn}a\right)F^{h(2)\prime}_{mn}\left(k^{h(2)}_{mn}a\right) \quad (6a)$$

$$\left((1+\delta_{m0})\Theta N^{h(2)}_{mn} k^{h(2)}_{mn}\frac{2}{\pi}F^{h(2)\prime}_{mn}\left(k^{h(2)}_{mn}a\right)\right)^2 \sum_p^\infty \left(N^{h(1)}_{mp}\frac{1}{\left(k^{h(1)}_{mp}\right)^2 - \left(k^{h(2)}_{mn}\right)^2}\right)^2$$
$$= \int_{A^{(1)}} \left|h^{(2)}_{zm}\right|^2 dS - \delta_{m0}\frac{4\Theta}{\left(\left(\frac{b}{a}\right)^2 - 1\right)\left(k^{h(2)}_{0n}\right)^2} \left(N^{h(2)}_{0n}F^{h(2)\prime}_{0n}\left(k^{h(2)}_{0n}a\right)\right)^2. \quad (6b)$$

If we choose the linear combination of (6a) and $-\left(k^{h(2)}_{mn}\right)^2$ times (6b) we get the closed-form expression:

$$-\delta_{m0}\frac{1}{(k_c a)^2}\frac{\pi^2}{4\Theta}\frac{1}{\left(\frac{b}{a}\right)^2 - 1} + \sum_p^\infty \left(N^{h(1)}_{mp}\right)^2 \frac{1}{\left(k^{h(1)}_{mp}a\right)^2 - (k^h_c a)^2}$$
$$= -\frac{1}{1+\delta_{m0}}\frac{1}{k^h_c a}\frac{\pi^2}{4\Theta}\frac{F^{h(2)}_{mn}(k^h_c a)}{F^{h(2)\prime}_{mn}(k^h_c a)}. \quad (7)$$

In (7), $k^{h(2)}_{mn}$ has been replaced by the wavenumber k^h_c. The integral

$$\int_{A^{(1)}} \left|h^{(2)}_{zm}\right|^2 dS$$

which appears in (6a) and in (6b) has no closed form. Fortunately, both integrals cancel each other.

The left-hand side of (7) is just the expression which has to be calculated in the GSD method. Now we are able to substitute the infinite sum by a closed-form expression. The inner radius c can completely be eliminated from (7) using the characteristic equation (5i) for waveguide (2).

B. TM Modes

Following a similar analysis for the TM modes we get by combining (4a) and (4c)

$$\sum_p^\infty \left(\left(N^{e(1)}_{mp}\right)^2 \frac{1}{\left(k^{e(1)}_{mp}a\right)^2 - (k^e_c a)^2} - \nu_m^2\left(N^{h(1)}_{mp}\right)^2 \frac{1}{\left(k^{h(1)}_{mp}a\right)^2 (k^e_c a)^2}\right)$$
$$= \frac{1}{k^e_c a}\frac{\pi^2}{4\Theta}\frac{F^{e(2)\prime}_{mn}(k^e_c a)}{F^{e(2)}_{mn}(k^e_c a)}. \quad (8)$$

The left-hand side of (8) represents that expression which the GSD method requires.

IV. Conclusion

Some closed-form expressions for infinite summations have been derived that are encountered in the application of the GSD method to structures containing sector waveguides. The numerical efficiency of the GSD method can consequently be increased by some orders of magnitude if the closed-form expressions are used.

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